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METHOD OF LIMIT RECURRENCE BY CONTRACTIVE MAPPING IN SOLVING A CLASS OF EXPONENTIAL KERNEL INTEGRAL EQUATIONS

A method of solving a class of exponential kernel integral equations is suggested, when a polynomialized term is added to the integral. The kernel is defined on the unit square. The analytical solutions are found using limit recurrence of the corresponding operator mapping which is contractive one. The unique fixed point (solution) is represented as the geometrical progression sum.
***Keywords*: operator mapping, contractive mapping, metric space, functional $[0;1]$ – defined space, operator equation, mapping kernel, limit recurrence, approximate solution, unique fixed point, unique solution, zeroth approximation, the n -th approximation, geometrical progression sum, $[0;1]$ – defined measurable functions, exponential kernel integral equation.**

Specifying the paper direction. It is well learned that the contractiveness [1] of operator mapping [2]

$$x(t) = \mathscr{A}\left[x(t)\right] \quad \forall\, x(t) \in \boldsymbol{X}_{\mathcal{M}} \text{ by } t \in \mathcal{M} \subset \mathbb{R} \tag{1}$$

lets solve the operator equation (1) approximately as recurrence

$$x_n(t) = \mathscr{A}\left[x_{n-1}(t)\right] \text{ for } n \in \mathbb{N} \text{ by } x_0(t) \in \boldsymbol{X}_{\mathcal{M}}. \tag{2}$$

The solution $x^*(t)$ of the operator equation (1) appears to be exact if the limit

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left(\mathscr{A}\left[x_{n-1}(t)\right]\right) \tag{3}$$

can be found analytically [3; 4]. However, sometimes it is pretty hard to determine this limit, except for some cases with progressions [3; 5].

Analysis of the availability. There may one even not be assured in $\rho_{\boldsymbol{X}_{\mathcal{M}}}$ -contractiveness of the mapping in (1), where

$$\rho_{\boldsymbol{X}_{\mathcal{M}}}\left(\mathscr{A}\left[x_1(t)\right], \mathscr{A}\left[x_2(t)\right]\right) \leqslant \alpha \rho_{\boldsymbol{X}_{\mathcal{M}}}\left(x_1(t), x_2(t)\right) \tag{4}$$

for the existing $\alpha \in (0;1)$ on the space $\boldsymbol{X}_{\mathcal{M}}$ metric $\rho_{\boldsymbol{X}_{\mathcal{M}}}$ $\forall\, x_1(t) \in \boldsymbol{X}_{\mathcal{M}}$ and $\forall\, x_2(t) \in \boldsymbol{X}_{\mathcal{M}}$, so here are no restrictions on applying the recurrence

(2). Certainly, the suspected solution $x^*(t)$ by the recurrence (2) should be checked by substituting $x^*(t)$ into the operator $\mathscr{A}$ sign.

An interest arises for integral equations

$$x(t) = f(t) + \int_0^1 K(t, s)x(s)ds \tag{5}$$

of the operator mapping (1) particular form [4; 6; 7], where $x(t) \in \boldsymbol{X}_{[0;1]}$ and the kernel $K(t, s) \in \boldsymbol{X}_{[0;1]} \times \boldsymbol{X}_{[0;1]}$ is defined on the unit square $[0; 1] \times [0; 1]$. The being added function $f(t) \in \boldsymbol{X}_{[0;1]}$ appears to be either a line or a polynomial [6]. And deeper particular, the kernel

$$K(t, s) = \psi\left(e^{t-s}\right) \tag{6}$$

is a linear function of the exponential e^{t-s} [8; 9].

Investigation goal. Having taken into consideration the kernel type (6), there is a paper goal to substantiate the exact solution

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left(f(t) + \int_0^1 K(t, s)x_{n-1}(s)ds \right) =$$

$$= \lim_{n\to\infty}\left(f(t) + \int_0^1 \psi\left(e^{t-s}\right)x_{n-1}(s)ds \right) \tag{7}$$

in the case

$$x(t) = f(t) + b\int_0^1 e^{t-s}x(s)ds\,, \tag{8}$$

where $b \in \mathbb{R}$ and the function $f(t) \in \boldsymbol{X}_{[0;1]}$ is polynomial. Firstly the equations

$$x(t) = a + b\int_0^1 e^{t-s}x(s)ds \tag{9}$$

and

$$x(t) = at + b\int_0^1 e^{t-s}x(s)ds \tag{10}$$

are going to be proved to have their exact solutions by some bounding the parameters $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Then, secondly, the generalizing them equation

$$x(t)=\sum_{i=0}^{p} a_i t^i + b\int_0^1 e^{t-s}x(s)ds \text{ by } p\in\mathbb{N}\cup\{0\} \quad (11)$$

should be proved to have its exact solution, where the number parameters $a_i\in\mathbb{R}$ $\forall\, i=\overline{0,\,p}$ and $b\in\mathbb{R}$ of the function $f(t)\in\boldsymbol{X}_{[0;\,1]}$ and the kernel $K(t,s)\in\boldsymbol{X}_{[0;\,1]}\times\boldsymbol{X}_{[0;\,1]}$ correspondingly must be bounded. It ought to be noted that the metric $\rho_{\boldsymbol{X}_{[0;\,1]}}$ of the space $\boldsymbol{X}_{[0;\,1]}$ is not specified here, as well as the space $\boldsymbol{X}_{[0;\,1]}$.

Solving exactly the integral equation (9). *Lemma 1.* The unique fixed point $x^*(t)\in\boldsymbol{X}_{[0;\,1]}$ in the mapping in (9) can be found as the limit (3), and the integral equation (9) solution is

$$x^*(t)=a+\frac{ab}{1-b}\left(e^t-e^{t-1}\right) \quad (12)$$

by $a\in\mathbb{R}\setminus\{0\}$ and $|b|<1$.

Proof. Though contractiveness of the mapping in (9) had not even been proved, there are no restrictions on applying the recurrence (3) as (7). Take the zeroth approximation

$$x_0(t)=0 \quad \forall\, t\in[0;\,1] \quad (13)$$

and will get

$$x_1(t)=a+b\int_0^1 e^{t-s}x_0(s)ds=a+b\int_0^1 e^{t-s}\cdot 0ds=a\,, \quad (14)$$

$$x_2(t)=a+b\int_0^1 e^{t-s}x_1(s)ds=a+b\int_0^1 e^{t-s}ads=$$

$$=a+abe^t\left(-e^{-s}\Big|_0^1\right)=a+abe^t\left(-e^{-1}+1\right)=a+ab\left(e^t-e^{t-1}\right), \quad (15)$$

$$x_3(t)=a+b\int_0^1 e^{t-s}x_2(s)ds=a+b\int_0^1 e^{t-s}\left[a+ab\left(e^s-e^{s-1}\right)\right]ds=$$

$$=a+abe^t\left(-e^{-s}\Big|_0^1\right)+ab^2e^t\left(1-e^{-1}\right)s\Big|_0^1=$$

$$=a+ab\left(e^t-e^{t-1}\right)+ab^2\left(e^t-e^{t-1}\right), \quad (16)$$

$$x_4(t)=a+b\int_0^1 e^{t-s}x_3(s)ds=a+b\int_0^1 e^{t-s}\left[a+ab\left(e^s-e^{s-1}\right)+ab^2\left(e^s-\right.\right.$$

$$-e^{s-1}\Big)\Big]ds = a + abe^{t}\left(-e^{-s}\Big|_0^1\right) + ab^2e^t\left(1-e^{-1}\right)s\Big|_0^1 + ab^3e^t\left(1-e^{-1}\right)s\Big|_0^1 =$$

$$= a + ab\left(e^t - e^{t-1}\right) + ab^2\left(e^t - e^{t-1}\right) + ab^3\left(e^t - e^{t-1}\right). \quad (17)$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the n -th approximation

$$x_n(t) = a + b\int_0^1 e^{t-s}x_{n-1}(s)ds = a + b\int_0^1 e^{t-s}\left(a + \sum_{k=1}^{n-2} ab^k\left(e^s - e^{s-1}\right)\right)ds =$$

$$= a + abe^t\left(-e^{-s}\Big|_0^1\right) + \sum_{k=1}^{n-2} ab^{k+1}e^t\left(1-e^{-1}\right)s\Big|_0^1 =$$

$$= a + ab\left(e^t - e^{t-1}\right) + \sum_{k=1}^{n-2} ab^{k+1}\left(e^t - e^{t-1}\right) =$$

$$= a + \sum_{l=1}^{n-1} ab^l\left(e^t - e^{t-1}\right) = a + ab\left(e^t - e^{t-1}\right)\left(1 + \sum_{k=1}^{n-2} b^k\right), \quad (18)$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[a + ab\left(e^t - e^{t-1}\right)\left(1 + \sum_{k=1}^{n-2} b^k\right)\right] =$$

$$= a + ab\left(e^t - e^{t-1}\right)\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2} b^k\right) = a + \frac{ab}{1-b}\left(e^t - e^{t-1}\right). \quad (19)$$

For avoiding triviality, the parameter $a \neq 0$, that is $a \in \mathbb{R} \setminus \{0\}$. Substituting $x^*(t)$ by (12) into the right member of (9) gives its left member exactly. The lemma has been proved.

Solving exactly the integral equation (10). *Lemma 2.* The unique fixed point $x^*(t) \in \boldsymbol{X}_{[0;1]}$ in the mapping in (10) can be found as the limit (3), and the integral equation (10) solution is

$$x^*(t) = at + \frac{ab}{1-b}\left(e^t - 2e^{t-1}\right) \quad (20)$$

by $a \in \mathbb{R} \setminus \{0\}$ and $|b| < 1$.

Proof. Though contractiveness of the mapping in (10) had not even been proved, there are no restrictions on applying the recurrence (3) as (7). Taking the zeroth approximation (13) gives

$$x_1(t) = at + b\int_0^1 e^{t-s}x_0(s)ds = at + b\int_0^1 e^{t-s}\cdot 0ds = at, \quad (21)$$

$$x_2(t) = at + b\int_0^1 e^{t-s} x_1(s)ds = at + b\int_0^1 e^{t-s} asds = at + abe^t \int_0^1 se^{-s} ds \quad (22)$$

Integrating the term

$$\int se^{-s} ds \quad (23)$$

by parts gives

$$\int se^{-s} ds = \int e^{-s} ds - se^{-s}. \quad (24)$$

Then, continuing (22),

$$x_2(t) = at + abe^t \left(\int_0^1 e^{-s} ds - se^{-s} \Big|_0^1 \right) = at + abe^t \left[\left(-e^{-s} \right) \Big|_0^1 - se^{-s} \Big|_0^1 \right] =$$

$$= at + abe^t \left(-e^{-1} + 1 - e^{-1} \right) = at + ab\left(e^t - 2e^{t-1} \right). \quad (25)$$

The third and fourth approximations are

$$x_3(t) = at + b\int_0^1 e^{t-s} x_2(s) ds =$$

$$= at + b\int_0^1 e^{t-s} \left[as + ab\left(e^s - 2e^{s-1} \right) \right] ds =$$

$$= at + abe^t \int_0^1 se^{-s} ds + ab^2 e^t \left(1 - 2e^{-1} \right) s \Big|_0^1 =$$

$$= at + ab\left(e^t - 2e^{t-1} \right) + ab^2 \left(e^t - 2e^{t-1} \right), \quad (26)$$

$$x_4(t) = at + b\int_0^1 e^{t-s} x_3(s) ds =$$

$$= at + b\int_0^1 e^{t-s} \left[as + ab\left(e^s - 2e^{s-1} \right) + ab^2 \left(e^s - 2e^{s-1} \right) \right] ds =$$

$$= at + abe^t \int_0^1 se^{-s} ds + ab^2 e^t \left(1 - 2e^{-1} \right) s \Big|_0^1 + ab^3 e^t \left(1 - 2e^{-1} \right) s \Big|_0^1 =$$

$$= at + ab\left(e^t - 2e^{t-1} \right) + ab^2 \left(e^t - 2e^{t-1} \right) + ab^3 \left(e^t - 2e^{t-1} \right). \quad (27)$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the n-th approximation

$$x_n(t) = at + b\int_0^1 e^{t-s} x_{n-1}(s) ds =$$

$$= at + b\int_0^1 e^{t-s} \left(as + \sum_{k=1}^{n-2} ab^k \left(e^s - 2e^{s-1} \right) \right) ds =$$

$$= at + abe^{t}\int_{0}^{1} se^{-s}ds + \sum_{k=1}^{n-2} ab^{k+1}e^{t}\left(1-2e^{-1}\right)s\Big|_{0}^{1} =$$

$$= at + ab\left(e^{t} - 2e^{t-1}\right) + \sum_{k=1}^{n-2} ab^{k+1}\left(e^{t} - 2e^{t-1}\right) =$$

$$= at + \sum_{l=1}^{n-1} ab^{l}\left(e^{t} - 2e^{t-1}\right) = at + ab\left(e^{t} - 2e^{t-1}\right)\left(1 + \sum_{k=1}^{n-2} b^{k}\right), \quad (28)$$

giving the geometrical progression in the last member sum. This progression is summable by $|b| < 1$ and then

$$x^{*}(t) = \lim_{n\to\infty} x_{n}(t) = \lim_{n\to\infty}\left[at + ab\left(e^{t} - 2e^{t-1}\right)\left(1 + \sum_{k=1}^{n-2} b^{k}\right)\right] =$$

$$= at + ab\left(e^{t} - 2e^{t-1}\right)\lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2} b^{k}\right) = at + \frac{ab}{1-b}\left(e^{t} - 2e^{t-1}\right). \quad (29)$$

For avoiding triviality, the parameter $a \neq 0$, that is $a \in \mathbb{R}\setminus\{0\}$. Substituting $x^{*}(t)$ by (20) into the right member of (10) gives its left member exactly. The lemma has been proved.

Solving exactly the integral equation (11), generalizing the equations (9) and (10) as given with polynomialized added term $f(t) \in \boldsymbol{X}_{[0;1]}$.

Theorem. The unique fixed point $x^{*}(t) \in \boldsymbol{X}_{[0;1]}$ in the mapping in (11) can be found as the limit (3), and

$$x^{*}(t) = \sum_{i=0}^{p} a_{i}t^{i} + \frac{be^{t}}{1-b}\sum_{i=0}^{p} a_{i}r(i) \quad (30)$$

is the solution of the integral equation (11), where

$$r(m) = mr(m-1) - \frac{1}{e},\ m \in \mathbb{N},\ r(0) = 1 - \frac{1}{e}, \quad (31)$$

by $a_{i} \in \mathbb{R}\ \forall\ i = \overline{0, p}$ in $\exists\ i_{0} \in \left\{\overline{0, p}\right\}$ with $a_{i_0} \in \mathbb{R}\setminus\{0\}$ and $|b| < 1$.

Proof. Though contractiveness of the mapping in (11) had not even been proved, there are no restrictions on applying the recurrence (3) as (7). Taking the zeroth approximation (13) gives

$$x_{1}(t) = \sum_{i=0}^{p} a_{i}t^{i} + b\int_{0}^{1} e^{t-s}x_{0}(s)ds = \sum_{i=0}^{p} a_{i}t^{i} + b\int_{0}^{1} e^{t-s}\cdot 0ds = \sum_{i=0}^{p} a_{i}t^{i}, \quad (32)$$

$$x_{2}(t) = \sum_{i=0}^{p} a_{i}t^{i} + b\int_{0}^{1} e^{t-s}x_{1}(s)ds = \sum_{i=0}^{p} a_{i}t^{i} + b\int_{0}^{1} e^{t-s}\left(\sum_{i=0}^{p} a_{i}s^{i}\right)ds =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \int_0^1 e^{-s} \left(\sum_{i=0}^{p} a_i s^i \right) ds = \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i \int_0^1 s^i e^{-s} ds \,. \quad (33)$$

Integrating the i-th term

$$\int s^i e^{-s} ds \quad (34)$$

within the nonlinear sum by parts gives

$$\int s^i e^{-s} ds = i \int s^{i-1} e^{-s} ds - s^i e^{-s} \quad \forall\, i = \overline{0,\, p} \text{ by } p \in \mathbb{N} \cup \{0\}\,. \quad (35)$$

Denoting

$$\int_0^1 s^i e^{-s} ds = i \int_0^1 s^{i-1} e^{-s} ds - \left(s^i e^{-s} \right)\Big|_0^1 =$$

$$= i \int_0^1 s^{i-1} e^{-s} ds - \frac{1}{e} = r(i) \quad \forall\, i = \overline{1,\, p} \text{ by } p \in \mathbb{N} \cup \{0\} \quad (36)$$

and having noted in (36) that

$$r(i) = ir(i-1) - \frac{1}{e} \text{ by } r(0) = 1 - \frac{1}{e}, \quad (37)$$

get the second approximation

$$x_2(t) = \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i \int_0^1 s^i e^{-s} ds = \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i r(i)\,. \quad (38)$$

The third and fourth approximations are

$$x_3(t) = \sum_{i=0}^{p} a_i t^i + b \int_0^1 e^{t-s} x_2(s) ds =$$

$$= \sum_{i=0}^{p} a_i t^i + b \int_0^1 e^{t-s} \left[\sum_{i=0}^{p} a_i s^i + be^s \sum_{i=0}^{p} a_i r(i) \right] ds =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i \int_0^1 s^i e^{-s} ds + \left(b^2 e^t \sum_{i=0}^{p} a_i r(i) \right) s \Big|_0^1 =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i r(i) + b^2 e^t \sum_{i=0}^{p} a_i r(i) =$$

$$= \sum_{i=0}^{p} a_i t^i + \left(be^t + b^2 e^t \right) \sum_{i=0}^{p} a_i r(i), \quad (39)$$

$$x_4(t) = \sum_{i=0}^{p} a_i t^i + b \int_0^1 e^{t-s} x_3(s) ds =$$

$$= \sum_{i=0}^{p} a_i t^i + b\int_0^1 e^{t-s}\left[\sum_{i=0}^{p} a_i s^i + be^s \sum_{i=0}^{p} a_i r(i) + b^2 e^s \sum_{i=0}^{p} a_i r(i)\right] ds =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i \int_0^1 s^i e^{-s} ds + \left(b^2 e^t \sum_{i=0}^{p} a_i r(i)\right) s\Big|_0^1 + \left(b^3 e^t \sum_{i=0}^{p} a_i r(i)\right) s\Big|_0^1 =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i r(i) + b^2 e^t \sum_{i=0}^{p} a_i r(i) + b^3 e^t \sum_{i=0}^{p} a_i r(i) =$$

$$= \sum_{i=0}^{p} a_i t^i + \left(be^t + b^2 e^t + b^3 e^t\right) \sum_{i=0}^{p} a_i r(i). \quad (40)$$

So, for $n \in \mathbb{N} \setminus \{1\}$ the n-th approximation

$$x_n(t) = \sum_{i=0}^{p} a_i t^i + b\int_0^1 e^{t-s} x_{n-1}(s) ds =$$

$$= \sum_{i=0}^{p} a_i t^i + b\int_0^1 e^{t-s}\left[\sum_{i=0}^{p} a_i s^i + \sum_{k=1}^{n-2} b^k e^s \sum_{i=0}^{p} a_i r(i)\right] ds =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i \int_0^1 s^i e^{-s} ds + \left(\sum_{k=1}^{n-2} b^{k+1} e^t \sum_{i=0}^{p} a_i r(i)\right) s\Big|_0^1 =$$

$$= \sum_{i=0}^{p} a_i t^i + be^t \sum_{i=0}^{p} a_i r(i) + \sum_{k=1}^{n-2} b^{k+1} e^t \sum_{i=0}^{p} a_i r(i) =$$

$$= \sum_{i=0}^{p} a_i t^i + \sum_{l=1}^{n-1} b^l e^t \sum_{i=0}^{p} a_i r(i) = \sum_{i=0}^{p} a_i t^i + be^t\left(1 + \sum_{k=1}^{n-2} b^k\right) \sum_{i=0}^{p} a_i r(i), \quad (41)$$

giving the geometrical progression in the last member sum by $k = \overline{1, n-2}$. This progression is summable by $|b| < 1$ and then

$$x^*(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\left[\sum_{i=0}^{p} a_i t^i + be^t\left(1 + \sum_{k=1}^{n-2} b^k\right)\sum_{i=0}^{p} a_i r(i)\right] = \sum_{i=0}^{p} a_i t^i +$$

$$+ \left[be^t \sum_{i=0}^{p} a_i r(i)\right] \lim_{n\to\infty}\left(1 + \sum_{k=1}^{n-2} b^k\right) = \sum_{i=0}^{p} a_i t^i + \frac{be^t}{1-b} \sum_{i=0}^{p} a_i r(i). \quad (42)$$

For avoiding triviality, the set of polynomial parameters $\{a_i\}_{i=0}^{p}$ should contain at least one element, that is $a_i \in \mathbb{R}$ $\forall\, i = \overline{0, p}$ in $\exists\, i_0 \in \left\{\overline{0, p}\right\}$ with $a_{i_0} \in \mathbb{R} \setminus \{0\}$. Substituting $x^*(t)$ by (30) with (31) into the right member of (11) gives its left member exactly. The theorem, generalizing the Lemma 1 and Lemma 2, has been proved.

Conclusion. After having proved the exponential kernel integral equations (9) — (11) to have their exact solutions (12), (20), and (30) with (31), they are passing to have been grouped in the table.

Table

The exact solutions of the exponential kernel integral equation (8), having applied its being contractive mapping generally

Integral equation (8) form by $a \in \mathbb{R} \setminus \{0\}$ and $a_i \in \mathbb{R}$ $\forall\, i = \overline{0, p}$ in $\exists\, i_0 \in \{\overline{0, p}\}$ with $a_{i_0} \in \mathbb{R} \setminus \{0\}$, $p \in \mathbb{N} \cup \{0\}$	Integral equation (8) form exact solution
$x(t) = a + b\int_0^1 e^{t-s} x(s)\,ds$	$x^*(t) = a + \frac{ab}{1-b}\left(e^t - e^{t-1}\right)$, $\lvert b \rvert < 1$
$x(t) = at + b\int_0^1 e^{t-s} x(s)\,ds$	$x^*(t) = at + \frac{ab}{1-b}\left(e^t - 2e^{t-1}\right)$, $\lvert b \rvert < 1$
$x(t) = \sum_{i=0}^{p} a_i t^i + b\int_0^1 e^{t-s} x(s)\,ds$	$x^*(t) = \sum_{i=0}^{p} a_i t^i + \frac{be^t}{1-b}\sum_{i=0}^{p} a_i r(i)$, $\lvert b \rvert < 1$, $r(m) = mr(m-1) - \frac{1}{e}$, $m \in \mathbb{N}$, $r(0) = 1 - \frac{1}{e}$

One may note that the solutions (12) and (20) might have been driven out from the solution (30) with (31).

Nevertheless, the particular cases with the solutions (12) and (20) are much convenient, than to deduce them from (30) with (31). And having substantiated the existence of the exact solution in the integral equation (11), had generalized the integral equations (9) and (10), the up-viewed table gives an indication to solve instantly a wide range of practical problems [4; 6; 10; 11], modeled as the mapping (1) kernel form (6), particularized as (9) — (11).

Certainly, the papered investigation may be proceeded towards generalizing the subset $\mathcal{M}$ of some $\mathbb{R}^N$ by $N \in \mathbb{N}$.

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МЕТОД ГРАНИЧНОЇ РЕКУРСІЇ У СТИСКАЮЧОМУ ВІДОБРАЖЕННІ ПРИ РОЗВ'ЯЗУВАННІ ОДНОГО КЛАСУ ІНТЕГРАЛЬНИХ РІВНЯНЬ З ЕКСПОНЕНЦІАЛЬНИМ ЯДРОМ

Пропонується метод розв'язання одного класу інтегральних рівнянь з експоненціальним ядром для випадку, коли до інтеграла додається поліноміалізований член. Ядро задається на одиничному квадраті. Аналітичні розв'язки знаходяться за допомогою викорис-

тання граничної рекурсії відповідного операторного відображення, котре є стискаючим. Єдина нерухома точка (розв'язок) представляється сумою геометричної прогресії.

***Ключові слова*: операторне відображення, стискаюче відображення, метричний простір, функціональний простір на $[0; 1]$, операторне рівняння, ядро відображення, гранична рекурсія, наближений розв'язок, єдина нерухома точка, єдиний розв'язок, нульове наближення, n -е наближення, сума геометричної прогресії, вимірні функції на $[0; 1]$, інтегральне рівняння з експоненціальним ядром.**

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МЕТОД ГРАНИЧНОЙ РЕКУРСИИ В СЖИМАЮЩЕМ ОТОБРАЖЕНИИ ПРИ РЕШЕНИИ ОДНОГО КЛАССА ИНТЕГРАЛЬНЫХ УРАВНЕНИЙ С ЭКСПОНЕНЦИАЛЬНЫМ ЯДРОМ

Предлагается метод решения одного класса интегральных уравнений с экспоненциальным ядром для случая, когда к интегралу прибавляется полиномиализированный член. Ядро задаётся на единичном квадрате. Аналитические решения находятся при использовании граничной рекурсии соответствующего операторного отображения, которое является сжимающим. Единственная неподвижная точка (решение) представляется суммой геометрической прогрессии.

***Ключевые слова*: операторное отображение, сжимающее отображение, метрическое пространство, функциональное пространство на $[0; 1]$, операторное уравнение, ядро отображения, граничная рекурсия, приближённое решение, единственная неподвижная точка, единственное решение, нулевое приближение, n -е приближение, сумма геометрической прогрессии, измеримые функции на $[0; 1]$, интегральное уравнение с экспоненциальным ядром.**
