ПРИКЛАДНА МАТЕМАТИКА

УДК 539.3 https://doi.org/10.31713/vt1201823

Smetankyna N. V., Senior Research Fellow, Doctor of Engineering, Postnyi O. V. (A. M. Podgorny Institute for Mechanical Engineering Problems of the National Academy of Sciences of Ukraine, Kharkiv)

MODELING OF NON-STATIONARY HEAT CONDUCTION IN LAYERED MEDIUM

A method for determination of thermal condition of layered medium is offered. Legendre polynomials are used as basis functions of the problem. The proposed approach allows to formalize an algorithm of the solution of a non-stationary heat conduction problem and to select a class of objects to which the given approach can be applied.
***Keywords*: layered medium, heat conduction, film heat source.**

1. Introduction

The majority of publications, devoted to thermal state of laminated structures, deal with deformation of such structures under conditions of steady temperature fields or dynamic temperature fields with prescribed distribution through the thickness [1]. The hypothesis about a piecewise-linear temperature distribution through the thickness of a laminated package is often applied [2]. However, the non-stationary character of a problem requires a more exact description of the temperature field obtained directly from solution of a heat conduction equation.

Methods for solving of non-stationary heat conduction problems are based on difference schemes [2]. Analytical approaches are used less commonly used analytical approaches [3]. For multilayer elements in structures, heat conduction problems are solved by involving different kinds of hypotheses on temperature distribution over the thickness of the pack of layers. The majority of papers use the following numerical computation methods: the finite difference method, the boundary elements method, and finite elements method.

Jane and Wu [4] used the Laplace transform and the finite difference method for solving dynamic and static thermal elasticity problems for multilayer conical shells. Using the finite elements method, Oguamanam et al. [5] studied the nonlinear response of a laminated symmetrical orthogonally reinforced cylindrical panel to sudden application of a heat flux. The panel is cantilevered to a hub with limited rotation around the central axis of rotation. The temperature field thickness is constant and changes exponentially with time. The system of nonlinear equations is solved with the Newton-Raphson method jointly with the Newmark integration method.

Diakoniuk and Savula [6] considered initially the boundary value problem of thermal conductivity in a multilayered medium with small layer thicknesses. For numerical investigation of the solution new semianalytic finite elements method is used.

The paper presents a method for solution of the one-dimensional non-stationary heat conduction problem in a laminated medium with an internal heat source based on introduction of the temperature distribution in each layer by a system of Legendre polynomials.

2. Statement of the problem

Let us consider a layered wall made up of I layers with constant thickness h_i $\left(i=\overline{1,I}\right)$. Convective heat transfer occurs on the top and bottom surfaces of the wall.

The non-stationary heat conduction equation and the boundary conditions for a multilayer layered medium are derived from the heat balance variational equation [7].

The heat equation for the i th layer has the form

$$\nu_i \Delta T_i = \frac{\partial T_i}{\partial t},\ i=\overline{1,n} \tag{1}$$

where $\nu_i = \lambda_i / \left(\gamma_i c_i\right)$ is the thermal diffusivity, λ_i is the thermal conductivity, γ_i is the density of the i th layer material, c_i is the specific heat at constant volume of the i th layer, T_i is temperature, t is time, n is the number of layers.

We take $T_i(0)=const$ as initial conditions. Besides, we suppose that the boundary condition of convective heat transfer over the top and bottom surfaces is the third kind boundary condition

$$\lambda_1 \nabla T_1 = \alpha_t\left(T_t - T_1\right),\ -\lambda_n \nabla T_n = \alpha_b\left[T_b - T_n(0,t)\right], \tag{2}$$

where α_t and α_b are the convective heat transfer coefficients and T_t and T_b are temperatures on the top and bottom surfaces, respectively.

The conditions of equality of heat flows and temperatures on interfaces of layers are:

$$-\lambda_i \nabla T_i\big|_{z_i=0} + \lambda_{i+1} \nabla T_{i+1}\big|_{z=h_{i+1}} = 0,$$

$$T_i\big|_{z_i=0} - T_{i+1}\big|_{z=h_{i+1}} = 0, \quad i=\overline{1,n-1}. \tag{3}$$

We assume that a heat-generating film with intensity P is placed between the first and the second layers. Then the condition of the heat flow transfer between the layers is

$$-\lambda_1 \nabla T_1 + \lambda_2 \nabla T_2 = P\,. \tag{4}$$

3. Method of the problem solution

We seek a solution for each layer in the form

$$T(z_i, t) = a(t) f_1(x) + b(t) f_2(x) + c(t) f_3(x)\,, \tag{5}$$

where $x = z_i / h_i$, $0 \le z_i \le h_i$, $0 \le x \le 1$. Coordinate z_i is measured from an internal surface of each layer.

As functions $f_k, k = 1, 2, 3$, we choose Legendre orthonormal polynomials

$$f_1 = 1, \quad f_2 = \sqrt{3}(2x-1), \quad f_3 = \sqrt{5}\left(6x^2 - 6x + 1\right), \tag{6}$$

$$\int_0^1 f_k f_l dx = \delta_{kl}\,. \tag{7}$$

Projecting the equation (1) to functions (3), we obtain

$$\frac{1}{h_i^2}\nu_i \int_0^1 \left[a_i(t) f_1''(x) + b_i(t) f_2''(x) + c_i(t) f_3''(x)\right] f_k(x) dx =$$
$$= \int_0^1 \left[\dot{a}_i(t) f_1(x) + \dot{b}_i(t) f_2(x) + \dot{c}_i(t) f_3(x)\right] f_k(x) dx \tag{8}$$

that in view of the condition (7) we get

$$\dot{a}_i(t) = \mu_i c_i(t), \quad \mu_i = 12\sqrt{5}\nu_i / h_i^2; \quad \dot{b}_i(t) = 0; \quad \dot{c}_i(t) = 0\,. \tag{9}$$

Taking into account (2) and (5), we obtain the following equalities

$$\frac{\lambda_1}{h_1}\left[2\sqrt{3} b_1(t) + 6\sqrt{5} c_1(t)\right] = \alpha_t \left[T_t - a_1(t) - \sqrt{3} b_1(t) - \sqrt{5} c_1(t)\right],$$
$$-\frac{\lambda_n}{h_n}\left[2\sqrt{3} b_n(t) - 6\sqrt{5} c_n(t)\right] = \alpha_b \left[T_b - a_n(t) + \sqrt{3} b_n(t) - \sqrt{5} c_n(t)\right]. \tag{10}$$

The conditions (3) and (4) take the form

$$a_i - \sqrt{3} b_i + \sqrt{5} c_i = a_{i+1} - \sqrt{3} b_{i+1} + \sqrt{5} c_{i+1}\,. \tag{11}$$

$$-\frac{\lambda_1}{h_1}\left[2\sqrt{3} b_1(t) - 6\sqrt{5} c_1(t)\right] + \frac{\lambda_2}{h_2}\left[2\sqrt{3} b_2(t) + 6\sqrt{5} c_2(t)\right] = P\,, \tag{12}$$

The unknown functions we find from conditions (10)-(12). We form a system of linear algebraic equations in coefficients $a_{ik}(t)$, $b_{ik}(t)$ and $c_{ik}(t)$

$$[\mathbf{S}]\mathbf{V} = \mathbf{Q}\,, \tag{13}$$

where

$$s_{11} = \alpha_t\,,\ s_{12} = \sqrt{3}\left(\alpha_t + 2\lambda_1 / h_1\right),\ s_{13} = \sqrt{5}\left(\alpha_t + 6\lambda_1 / h_1\right),$$

$$s_{22} = -2\sqrt{3}\lambda_1/h_1 \,,\; s_{23} = 6\sqrt{5}\lambda_1/h_1 \,,\; s_{24} = 2\sqrt{3}\lambda_2/h_2 \,,\; s_{25} = 6\sqrt{5}\lambda_2/h_2 \,,$$
$$s_{31} = 1 \,,\; s_{32} = -\sqrt{3} \,,\; s_{33} = \sqrt{5} \,,\; s_{34} = -1 \,,\; s_{35} = -\sqrt{3} \,,\; s_{36} = -\sqrt{5} \,,$$
$$s_{45} = -2\sqrt{3}\lambda_2/h_2 \,,\; s_{46} = 6\sqrt{5}\lambda_2/h_2 \,,\; s_{48} = 2\sqrt{3}\lambda_3/h_3 \,,$$
$$s_{49} = 6\sqrt{5}\lambda_3/h_3 \,,\; s_{54} = 1 \,,\; s_{52} = -\sqrt{3} \,,\; s_{53} = \sqrt{5} \,,$$
$$s_{54} = -1 \,,\; s_{55} = -\sqrt{3} \,,\; s_{56} = -\sqrt{5} \,,\; \ldots \,,$$
$$s_{2n-2,3n-4} = -2\sqrt{3}\lambda_{n-1}/h_{n-1} \,,\; s_{2n-2,3n-3} = 6\sqrt{5}\lambda_{n-1}/h_{n-1} \,,\; s_{2n-2,3n-1} = 2\sqrt{3}\lambda_n/h_n \,,$$
$$s_{2n-2,n} = 6\sqrt{5}\lambda_n/h_n \,,\; s_{2n-1,3n-5} = 1 \,,\; s_{2n-1,3n-4} = -\sqrt{3} \,,\; s_{2n-1,3n-3} = \sqrt{5} \,,$$
$$s_{2n-1,3n-2} = -1 \,,\; s_{2n-1,3n-1} = -\sqrt{3} \,,\; s_{2n-1,3n} = -\sqrt{5} \,,\; s_{2n,3n-2} = \alpha_b \,,$$
$$s_{2n,3n-1} = -\sqrt{3}\left(\alpha_b + 2\lambda_n/h_n\right),\; s_{2n,3n} = \sqrt{5}\left(\alpha_b + 6\lambda_n/h_n\right),$$
$$q_1 = \alpha_t T_t \,,\; q_2 = P \,,\; q_{2n} = \alpha_b T_b \,,$$
$$v_1 = a_1 \,,\; v_2 = b_1 \,,\; v_3 = c_1 \,,\; \ldots \,,\; v_{n-2} = a_n \,,\; v_{n-1} = b_n \,,\; v_n = c_n \,.$$

The remaining components of the matrix and the right side vector are zero. The solution obtained in this case at each time step will be exactly satisfy the boundary conditions and conditions on interfaces of layers. Non-stationary feature will be reflected in the fact that the functions $a_i(t)$ we determine from the Cauchy problem.

The matrix of system (13) $[\mathbf{S}]$ has $2n$ rows and $3n$ columns. We express the functions $b_i(t)$ and $c_i(t)$ through $a_i(t)$ and the vector **Q**. We transfer coefficients at $a_i(t)$ to the right side and form the matrix $[\mathbf{B}]$. We denote the matrix in the left side as $[\mathbf{A}]$. It has the order $2n \times 2n$.

Then we get the system

$$[\mathbf{A}]\mathbf{Y} = [\mathbf{B}]\mathbf{X} + \mathbf{Q} \tag{14}$$

where **Y** is vector with components $b_i(t)$, $c_i(t)$ $(i = 1, \ldots, n)$, and **X** is vector of coefficients $a_i(t)$.

We write the solution of the system (14) as

$$\mathbf{Y} = [\mathbf{A}]^{-1}[\mathbf{B}]\mathbf{X} + [\mathbf{A}]^{-1}\mathbf{Q} \,.$$

Then we solve this system of differential equations by a modified method of expanding the solution into a Taylor series [8; 9].

4. Conclusion

A method for solution of non-stationary heat conduction problem in layered wall is proposed. The transient temperature change is caused by an impulse action of a distributed heat source simulating a heat-generating film. Temperature distribution through the thickness of each layer is represented by using Legendre orthonormal polynomials, which allows authentic description of the thermal condition of layered elements assembled from layers with different mechanical and geometrical characteristics.

The solution of such problems has practical importance, as the results of this research can be applied, for example, to the analysis of the efficiency of heating systems.

1. Goldstein R. J. Heat transfer – a review of 2003 literature / R. J. Goldstein, W. E. Ibele, S. V. Patankar // Int. J. Heat Mass Transfer. 2006. Vol. 49, N 3-4. P. 451–534. **2.** Shih T.-M. Literature survey of numerical heat transfer (2010–2011) / T.-M. Shih, Y. Zheng, M. Arie, J.-Ch. Zheng // Numerical Heat Transfer. Part A : Applications. 2013. Vol. 64, Iss. 6. P. 435–525. **3.** Kayhani M. H. A general analytical solution for heat conduction in cylindrical multilayer composite laminates / M. H. Kayhani, M. Norouzi, A. A. Delouei // Int. J. Thermal Sciences. 2012. Vol. 52. P. 73–82. **4.** Jane K. C. A generalized thermoelasticity problem of multilayered conical shells / K. C. Jane, Y. H. Wu // Int. J. Solids and Struct. 2004. Vol. 41, N 9–10. P. 2205–2233. **5.** Oguamanam D.C.D. Nonlinear transient response of thermally loaded laminated panels / D.C.D. Oguamanam, J. S. Hansen, G. R. Heppler // Trans. ASME. J. Appl. Mech. 2004. V. 71, № 1. P. 49–56. **6.** Diakoniuk L. M. Heterogeneous model of the process of thermal conductivity in a multilayered medium with thin layers / L. M. Diakoniuk, Ya. H. Savula // Journal of Numerical & Applied Mathematics. 2017. N 3 (126). P. 33–47. **7.** Altenbach I. Die methode der finiten elemente in der festkörpermechanik / I. Altenbach, U. Gabbert, J. Dankert, H. Köppler, S. Koczyk, A. S. Sacharov, V. N. Kislookij, V. V. Kiričevskij. Leipzig : VEB Fachbuchverlag, 1982. 480 p. **8.** Shupikov A. N. Nonstationary heat conduction in complex-shape laminated plates / A. N. Shupikov, N. V. Smetankina, Ye. V. Svet // Trans. ASME. J. Heat Transfer. 2007. V. 129, № 3. P. 335–341. **9.** Smetankina N. V. Analysis of temperature fields in laminated plates with film heat sources / N. V. Smetankina // Thesis of XVI International Conference Reports "Dynamical System Modelling and Stability Investigation". 2013. P. 321.

REFERENCES:

1. Goldstein R. J. Heat transfer – a review of 2003 literature / R. J. Goldstein, W. E. Ibele, S. V. Patankar // Int. J. Heat Mass Transfer. 2006. Vol. 49, N 3-4. P. 451–534. **2.** Shih T.-M. Literature survey of numerical heat transfer (2010–2011) / T.-M. Shih, Y. Zheng, M. Arie, J.-Ch. Zheng // Numerical Heat Transfer. Part A : Applications. 2013. Vol. 64, Iss. 6. P. 435–525. **3.** Kayhani M. H. A general analytical solution for heat conduction in cylindrical multilayer composite laminates / M. H. Kayhani, M. Norouzi, A. A. Delouei // Int. J. Thermal Sciences. 2012. Vol. 52. P. 73–82. **4.** Jane K. C. A generalized thermoelasticity problem of multilayered conical shells / K. C. Jane, Y. H. Wu // Int. J. Solids and Struct. 2004. Vol. 41, N 9–10. P. 2205–2233. **5.** Oguamanam D.C.D. Nonlinear transient response of thermally loaded laminated panels / D.C.D. Oguamanam, J. S. Hansen, G. R. Heppler // Trans. ASME. J. Appl. Mech. 2004. V. 71, № 1. P. 49–56. **6.** Diakoniuk L. M. Heterogeneous model of the process of thermal conductivity in a multilayered medium with thin layers / L. M. Diakoniuk, Ya. H. Savula // Journal of Numerical & Applied Mathematics. 2017. N 3 (126). P. 33–47. **7.**

Altenbach I. Die methode der finiten elemente in der festkörpermechanik / I. Altenbach, U. Gabbert, J. Dankert, H. Köppler, S. Koczyk, A. S. Sacharov, V. N. Kislookij, V. V. Kiričevskij. Leipzig : VEB Fachbuchverlag, 1982. 480 p. **8.** Shupikov A. N. Nonstationary heat conduction in complex-shape laminated plates / A. N. Shupikov, N. V. Smetankina, Ye. V. Svet // Trans. ASME. J. Heat Transfer. 2007. V. 129, № 3. P. 335–341. **9.** Smetankina N. V. Analysis of temperature fields in laminated plates with film heat sources / N. V. Smetankina // Thesis of XVI International Conference Reports “Dynamical System Modelling and Stability Investigation”. 2013. P. 321.

Рецензент: д.т.н., професор Сафоник А. П. (НУВГП)

Сметанкина Н. В., с.н.с., д.т.н., Постний О. В. (Інститут проблем машинобудування ім. А.М. Підгорного НАН України, м. Харків)

МОДЕЛЮВАННЯ НЕСТАЦІОНАРНОЇ ТЕПЛОПРОВІДНОСТІ ШАРУВАТОГО СЕРЕДОВИЩА

Запропоновано метод визначення теплового стану шаруватого середовища. Як базисні функції задачі для кожного шару використовуються поліноми Лежандра. Розроблений підхід дає можливість формалізувати алгоритм розв'язання нестаціонарної задачі теплопровідності і виділити клас об'єктів, для яких даний підхід можна застосувати.
***Ключові слова*: шарувате середовище, теплопровідність, плівкове джерело тепла.**

Сметанкина Н. В., старший научный сотрудник, доктор технических наук, Постный А. В. (Институт проблем машиностроения им. А.Н. Подгорного НАН Украины, Харьков)

МОДЕЛИРОВАНИЕ НЕСТАЦИОНАРНОЙ ТЕПЛОПРОВОДНОСТИ СЛОИСТОЙ СРЕДЫ

Предложен метод определения теплового состояния слоистой среды. Как базисные функции задачи для каждого слоя используются полиномы Лежандра. Разработанный подход дает возможность формализовать алгоритм решения нестационарной задачи теплопроводности и выделить класс объектов, для которых данный подход можно применить.
***Ключевые слова*: слоистая среда, теплопроводность, пленочный источник тепла.**
